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Convergence of Positive Operators

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In this paper we discuss two problems connected with Korovkin sets for sequences of positive linear operators.

1. MINIMAL KOROVKIN SETS ON SPHERES

Let X be a compact metric space, and let $C(X)$ be the space of all real continuous functions on X. We denote by $S = \{g_0, g_1, \ldots, g_m\}$ a finite subset of $C(X)$ which consists of linearly independent functions, and by $G = \operatorname{lin} S$ the linear hull of S. The *shadow* $\mathrm{S}(G)$ of G (or of S) is the set of all functions $f \in C(X)$ with the following property. If T_n, $n = 1, 2, \ldots$, is a sequence of positive linear operators mapping $C(X)$ into itself, and if $T_n g \to g$ for all $g \in G$, then $T_n f \to f$.

The set S (or G) is a *Korovkin set* if $S(G)$ is the whole space, that is, if the above implication holds for all $f \in C(X)$. In particular, this is so (and then S is called a *strict Korovkin set*) if each point $x_0 \in X$ is a peak point of G. We call $x_0 \in X$ a *peak point* of a linear subspace G of $C(X)$ if there is a function g_0 in the closure $\bar{G}$ of G for which $g_0(x_0) = 0$, $g_0(x) > 0$ for $x \neq x_0$.

In his important paper [11], Šaškin has given a "geometric" theory of Korovkin sets based on the following map Φ of X into R^{m+1}:

$$\Phi\colon x \to y = (g_0(x), g_1(x), \ldots, g_m(x)), \tag{1.1}$$

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and on the properties of extreme points of convex sets. In particular, it follows from his theory (see Klimov, Krasnosl'skiĭ and Lifšič [7, p. 68]) that a Korovkin set on the unit sphere S_{n-1} in R^n has at least $n+1$ elements. This lower bound is achieved by the trivial Korovkin set $1, y_1, ..., y_n$ on S_{n-1}, where the y_k are the cartesian coordinates in R^n: this is a *minimal Korovkin set* on S_{n-1}. We shall characterize all such sets.

A set of continuous functions $S = \{g_0, ..., g_m\}$ on X will be called *2-independent*, if no 3 points of X have images under Φ which lie on a two-dimensional plane through the origin. Under this assumption, Φ is a homeomorphism. We call a set Y in R^{m+1} *2-independent*, if no 3 points of Y lie on a plane of this type. Korovkin sets are 2-independent [11].

THEOREM 1. *A 2-independent set of functions $S = \{g_0, ..., g_m\}$ on the sphere S_{m-1} is a strict Korovkin set with respect to sequences of positive linear operators on $C(S_{m-1})$.*

This theorem (given also in [8]) admits a simple geometric interpretation. By means of the map Φ the statement translates: The set of coordinates $\{y_0, ..., y_m\}$ is a strict Korovkin set on $Y = \Phi(S_{m-1})$. As a geometric formulation of Theorem 1 we obtain therefore:

THEOREM 2. *Let Y be a topological image of S_{m-1} in R^{m+1}, which is 2-independent. Then at each point $y^{(0)}$ of Y there exists a strict supporting hyperplane for Y, passing through the origin.*

Proof. Each two-dimensional plane through the origin intersects Y at most twice. In particular, $0 \notin Y$, and each straight line through the origin has at most one point on Y. Let Σ be the unit sphere of R^{m+1} with center 0. The sets Y and $-Y$ are disjoint, they are projected onto Σ by rays through the origin into two topological images of S_{m-1}, say Σ' and $-\Sigma'$. The sphere Σ' divides Σ into two closed regions. Let B be the one of them which contains no points of $-\Sigma'$. By symmetry, $-\Sigma'$ is the boundary of $-B$, disjoint with B.

Let now $y^{(0)}$ be an arbitrary interior point of B. We consider any two dimensional plane through $y^{(0)}$ and 0, and the one-dimensional circle σ-intersection of the plane with Σ. We wish to prove that $\sigma \cap B$ is a circular arc $y^{(1)}y^{(2)}$ of an opening $< \pi$, with $y^{(1)}, y^{(2)} \in \Sigma'$, all other points being interior points of B.

In fact, there are on σ interior points of $\sigma \cap B$, for example $y^{(0)}$, and points not belonging to B, for example $-y^{(0)}$. Hence there are at least two, and since Y is 2-independent, exactly two points $y^{(1)}$, $y^{(2)}$ of Y on σ. Let $y^{(1)}y^{(2)}$ be the arc containing $y^{(0)}$. Then it coincides with $\sigma \cap B$, and since $y^{(2)} \notin -B$, has an opening $< \pi$ (see Fig. 1).

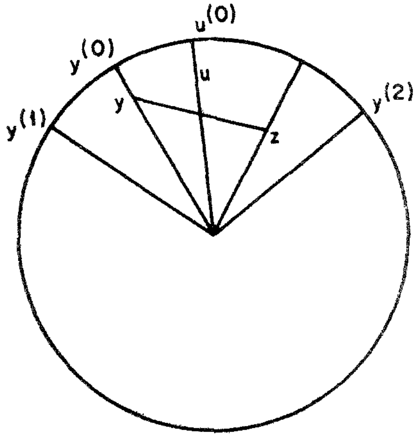

FIGURE 1

Let K be the closed cone formed by the rays connecting 0 with all points of B. It is easy to prove that the boundary of K consists of all rays connecting 0 with Y (or with Σ'). We show that K is convex.

Assume that $y, z \in K$ and that $u = \frac{1}{2}(y + z) \in K$. Without loss of generality, we may assume that the line yz does not go through 0, in particular that $y \neq 0$, $z \neq 0$, and further that the projection $y^{(0)}$ of y onto Σ is an interior point of B. The line yz and 0 span a plane, and the configuration of Fig. 1 applies. It follows that $u^{(0)} \in B$, a contradiction.

Let $y \in Y$, and let H be a supporting hyperplane for K at y. Assume that some other point $z \in Y$, $z \neq y$ belongs to H. Then the segment yz is contained in $K \cap H$, and hence it belongs to the boundary of K. If $u' = \frac{1}{2}(y + z)$, there is a point $u \in Y$ on the ray $0u'$. This is a contradiction: the plane through z, y, and 0 contains three distinct points of Y. Thus, H strictly supports Y.

In [11], Šaškin has a more general assertion than our Thoerem 1. However, the proof given there [11, p. 140, lines 1–8] is apparently incorrect, in particular when $m \geqslant 3$.

2. THE ROLE OF QUASI PEAK POINTS

From now on, G is an arbitrary subspace of $C(X)$; without loss of generality, we assume that G is closed. For the determination of the shadow of G, the notion of Choquet boundary is important. Useful tools for this purpose (see Berens and Lorentz [4]) are the upper and the lower envelopes $\underline{f}$, $\bar{f}$ of H. Bauer ([1], [2]) and Baskakov [3] of a function $f \in C(X)$:

$$\underline{f}(x) = -(\overline{-f})(x), \qquad \bar{f}(x) = \inf\{g(x): g \in G, g \geqslant f\}. \qquad (2.1)$$

The envelopes $\underline{f}$, $\bar{f}$ are defined if G contains a strictly positive function: $g_0(x) \geqslant c > 0$, $x \in X$. But beyond this, in the determination of the shadow, it is unnecessary and unnatural to assume that G contains the function 1 or that the functions $g \in G$ separate points of X. (These assumptions are usually made when defining the Choquet boundary $\partial_{\mathrm{Ch}} G$.) Hence the following definition, a generalization of the notion of the Choquet boundary. The *boundary* ∂G of G is the set of all $x_0 \in X$ with the following property: *All positive linear extensions to* $C(X)$ *of the restriction to* G *of the evaluation functional* $\epsilon_{x_0}(f) = f(x_0)$, $f \in C(X)$ *coincide on* $C(X)$. One shows ([1, 2, 4]) that this happens exactly when the upper and the lower envelopes (2.1) coincide at x_0: $\underline{f}(x_0) = \bar{f}(x_0)$, $f \in C(X)$. On the other hand, (Baskakov [3], Berens and Lorentz [4]), the shadow $\mathrm{S}(G)$ proves to be equal to the set of all f for which $\underline{f}(x) = \bar{f}(x)$ for all $x \in X$. As a corollary, G is a Korovkin set if and only if $\partial G = X$.

Korovkin sets, shadows, and boundaries have been often described in terms of peak points. We would like to observe that *quasi peak points* often provide a better description.

DEFINITION 1. A point $x_0 \in X$ is a quasi peak point for the space G with respect to the class T^+ of positive linear operators, if for each ϵ, $0 < \epsilon < 1$, and each neighborhood U of x_0, there exists a $g \in G$ with the properties:

$$g(x) \geqslant 0, \qquad x \in X \tag{2.2}$$

$$g(x_0) < \epsilon \tag{2.3}$$

$$g(x) \geqslant 1 \qquad \text{for} \quad x \in X \backslash U. \tag{2.4}$$

THEOREM 3. *Assume that the space* G *contains a strictly positive function* g_0. *Then the boundary* ∂G *of* G *is equal to the set of all quasi peak points of* G.

This is precisely what is needed for convergence theorems of Korovkin's type. In Bishop and de Leeuw [5] and in [10] the theorem is proved when G is a closed subalgebra of $C(X)$ which contains the function 1 and separates points of X. Compare also [6].

Proof. We can show that the following statements are equivalent for $x_0 \in X$:

(i) The restriction of the evaluation functional ϵ_{x_0} to G has a unique positive linear extension from G to $C(X)$.

(ii) $\underline{f}(x_0) = \bar{f}(x_0)$ for all $f \in C(X)$.

(iii) x_0 is a quasi peak point of G.

For the equivalence of (i) and (ii), see [4]. If (ii) holds, we select a function $f \geqslant 0$, for which $f(x_0) = 0$, $f(x) \geqslant 1$ for $x \in X \backslash U$. The existence of a $g \in G$ satisfying (2.2)–(2.4) follows from the definition (2.1).

Conversely, we derive (i) from (iii). Let L be a positive linear functional coinciding with ϵ_{x_0} on G, and let μ be the corresponding measure, so that $L(f) = \int_X f\, d\mu$, $f \in C(X)$. If g satisfies (2.2)–(2.4), then

$$\mu(X \backslash U) \leqslant \int_X g\, d\mu = L(g) = g(x_0) < \epsilon.$$

Thus $\mu(X \backslash U) = 0$, and since U is arbitrary, the whole mass of μ is concentrated at x_0. Hence

$$L(f) = \int_X f\, d\mu = f(x_0).$$

COROLLARY. *A subspace G of $C(X)$, containing a strictly positive function, is a Korovkin set if and only if each $x \in X$ is a quasi peak point.*

If G contains a g_0, and h is another strictly positive function on $C(X)$, then the space G and G', given by

$$G' = \{g' : g' = g/h, g \in G\}, \tag{2.6}$$

have similar properties. It is clear that G and G' have the same peak points. In addition we have:

$$\partial G' = \partial G. \tag{2.7}$$

Indeed, let $x_0 \in \partial G$, then the only positive linear functional L on $C(X)$ which satisfies $L(g) = g(x_0)$, is the evaluation functional ϵ_{x_0}. Let $L' \geqslant 0$ be a positive linear functional for which $L'(g') = g'(x_0)$, $g' \in G'$. Then $L_1(f) = L'(f/h)\, h(x_0)$ is also positive and linear and $L_1(g) = g(x_0)$, $g \in G$, hence $L_1 = \epsilon_{x_0}$ and $L' = \epsilon_{x_0}$, so that $x_0 \in \partial G'$.

In particular, we may take $h = g_0$, then G' contains the function 1.

THEOREM 4. *If G contains a strictly positive function g_0, then the set of all peak points of G is dense in ∂G.*

If G contains the function 1 and separates points of X, this assertion is Mil'man's theorem [9], also reproduced in Phelps [10, p. 57].

Proof. Without loss of generality, we may assume that G contains the function 1. Let $x_0 \in \partial G$, let $0 < \epsilon < 1$, and let U be a neighborhood of x_0. We select a function $g \in G$ satisfying (2.2)–(2.4). Subtracting from g, if necessary, a small constant, we may assume that g vanishes at some point

$x_1 \in U$. Then $1 - g/\| g \|$ is positive and belongs to the unit sphere of the space G. By a theorem of Mazur [10, p. 55], our function can be approximated by a *smooth function* g_1 of the unit sphere of G. But it is not difficult to see that smooth functions in G are precisely the functions whose absolute value admits just one maximum equal to one. In the present case, $g_1(x)$ itself must be equal to one at some point $x_2 \in U$, and else satisfy $-1 < g_1(x) < 1$. Then $1 - g_1(x)$ will have its minimum on X equal to zero, and the minimum will be attained only at x_2.

3. Quasi Peak Points for Positive Contractions

Similar arguments apply to the class $\mathbf{T}_1^+$ of positive linear operators of norm $\leqslant 1$ which map $C(X)$ into itself. About the subspace $G \subset C(X)$ we assume this time only that it is closed and *nonvanishing*: for each $x \in X$ there is a $g \in G$ with $g(x) \neq 0$. We shall make use of the functionals defined by M. Donner for some other purpose,

$$\bar{\bar{f}}(x_0) = \inf\{\lambda + g(x_0): \lambda \geqslant 0, g \in G, \lambda + g \geqslant f\}, \tag{3.1}$$

$$\underline{\underline{f}}(x_0) = -(\overline{\overline{-f}})(x_0). \tag{3.2}$$

Lemma 1. *Let L be a linear bounded functional on $C(X)$, let $x_0 \in X$ be given. Then L is a positive, of norm one extension of the restriction of ϵ_{x_0} from G to $C(X)$ if and only if*

$$L(f) \leqslant \bar{\bar{f}}(x_0), \qquad f \in C(X). \tag{3.3}$$

Lemma 2. *For a function $f_0 \in C(X)$, the value $L(f_0)$ of the above extension of ϵ_{x_0} is detremined uniquely if and only if*

$$\underline{\underline{f}}_0(x_0) = \bar{\bar{f}}_0(x_0). \tag{3.4}$$

The easy proofs, similar to those for $\bar{f}$ in [4], are left to the reader.

Definition 2. A point $x_0 \in X$ is a peak point for G with respect to $\mathbf{T}_1^+$ if there exists a function $g \in G$ which attains at x_0 a nonnegative strict maximum λ:

$$g(x_0) = \lambda, \qquad g(x) < \lambda, \qquad x \neq x_0. \tag{3.5}$$

DEFINITION 3. A point $x_0 \in X$ is a quasi peak point for G with respect to T_1^+ if for each $\epsilon > 0$ and each neighborhood U of x_0, there exists a $\lambda \geqslant 0$ and a $g \in G$ with the properties

$$g \leqslant \lambda \tag{3.6}$$

$$g(x_0) > \lambda - \epsilon \tag{3.7}$$

$$g(x) \leqslant \lambda - 1, \qquad x \in X \backslash U. \tag{3.8}$$

In the situation (3.5), $g(x) \leqslant \lambda - \delta$, $x \in X \backslash U$ for some $\delta > 0$. Replacing g by g/δ and λ by λ/δ, we obtain (3.6)–(3.8). It follows that each peak point is also a quasi peak point.

THEOREM 5. *Let G be a nonvanishing subspace of $C(X)$ and let $x_0 \in X$. The evaluation functional ϵ_{x_0} on G has a unique positive extension of norm one if and only if x_0 is a quasi peak point with respect to T_1^+.*

Proof. (a) Let the extension be unique. By Lemma 2, $f(x_0) = \bar{\bar{f}}(x_0)$ for each $f \in C(X)$. If $\epsilon > 0$, U are given, let f be such that $f \geqslant 0$, $f(x_0) = 0$ and $f(x) \geqslant 1$ outside of U. From the definition of $\bar{\bar{f}}$ we obtain a pair λ, g with the properties $\lambda + g \geqslant 0$, $\lambda + g(x_0) < \epsilon$ and $\lambda + g(x) \geqslant 1$, $x \in X \backslash U$.

(b) Conversely, let (3.6)–(3.8) be satisfied. Let L be a positive functional of norm 1 (or $\leqslant 1$) which extends ϵ_{x_0} from G. If μ is the corresponding measure,

$$\mu(X \backslash U) \leqslant \int_X (\lambda - g)\, d\mu = \lambda \mu X - L(g) \leqslant \lambda - g(x_0) < \epsilon.$$

The conclusion is as before.

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